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Impulsive pinning control algorithm of stochastic multi-agent systems with unbounded distributed delays

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Abstract The mean square exponential consensus problem for stochastic multi-agent systems with unbounded distributed delays is investigated. Two delayed impulsive pinning control protocols are proposed that are different from those of existing delay-free impulsive pinning controllers. By employing the Lyapunov function method, sufficient conditions are constructed under the presented strategies. It is shown that the system can realize mean square exponential consensus by controlling a small portion of agents. Then, the simulations indicate that the scheme is both feasible and effective.

Keywords Consensus · Delayed impulse · Unbounded distributed delays · Impulsive pinning control

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1 Introduction

Recently, consensus problems in multi-agent systems have aroused great interest in multidisciplinary research areas. These dynamical systems have widespread applications in several fields including power networks, social networks and so on [1–4]. The term consensus implies that all agents achieve agreement on some common values by resorting to suitable control strategies. In [5], the control protocol was designed to ensure tracking consensus of leader-following systems using local information.

However, in real applications, there are several interference factors, such as noises, external disturbance, communication delays and model uncertainty. Recently, research on consensus for stochastic multi-agent systems has been fruitful, which can be referred to Refs. [6–9]. On the other hand, owing to networks congestion and limited communication bandwidth, time delays are inevitable and may result in deterioration of control performance or even system instability. There are several methods for handling delayed systems, for example, Lyapunov functions, Krasovskii functionals, Halanay inequality, and so on. Consensus problems for delayed multi-agent systems were investigated in [10]. Wu et al. [11] provided a careful analysis of sampling synchronization in delayed networks with Markovian jumps. Applying the input delay approach, two delay-dependent criteria were obtained whereby synchronization of the master-slave systems was achieved. It is known that signal propagation is instantaneous;

thus, it can be modeled using distributed time delays. Therefore, the distributed time delays should be taken into consideration in the research. Moreover, the combination of discrete time delays and distributed time delays has been considered in [12–14]. However, in most of the literature, the delay kernel should be one. Otherwise, Jensen's inequality would not apply. Stability with unbounded distributed delays was investigated by an algebraic approach in [15–17], and relatively weak conditions were obtained. Based on the LMI approach, Yang et al. [16] studied synchronization problem for dynamics networks with unbounded distributed delays. It should be noted that research on the consensus problem with discrete delays and unbounded distributed delays has been scarce. Impulsive control is an effective control strategy for dealing with consensus problems, as it has the following advantages: (1) it uses only small control impulses at discrete instants and can thus save the energy and reduce control cost; (2) it provides an useful means to solve multi-agent systems that cannot tolerate continuous interference. As impulsive control involves instantaneous jumps at some time, transmission energy can be reduced. In Refs. [18,19], novel impulsive control strategies were applied to quasi-synchronization and exponential synchronization, and certain criteria were established. Multi-agent systems are generally consisting of many agents, it may be costly or even impossible to control every agent in a network by adding the controllers. To resolve this, pinning control was proposed. The principle of pinning control is to control the entire network by pinning a small number of agents. Therefore, control primarily acts on certain pinned agents and is then transmitted to unpinned agents through communication interconnection. If impulsive control and pinning control are integrated, control energy is significantly reduced. Thus, the impulsive pinning control strategies are proposed. Recently, in [20–22], impulsive pinning control was investigated. In this strategy, impulsive control is applied only to a small number of pinned agents, thus further reducing data communication load and greatly improving the efficiency of the communication channel. These studies generally considered delay-free impulsive pinning control only. As network-induced and computational delays cannot be ignored, delayed impulsive pinning control strategy was proposed in [23,24]. Stabilization analysis for neural networks was discussed in [23], where delayed impulsive pinning control was proposed. However, results regarding delayed impulses are scarce [25–27]. Owing to the effectiveness of impulsive pinning control and the extensive presence of time delays, it is important to investigate the delayed cases.

To the best of the author's knowledge, no relevant research on the delayed impulsive pinning control strategy of stochastic multi-agent systems with discrete time delays and unbounded distributed delays has been conducted. Based on the previous work, we try to fill this gap. Using Lyapunov stability theory for impulsive control systems, an effective impulsive pinning control algorithm is proposed for ensuring mean square exponential consensus. The major contributions are the following: (1) The model: a stochastic multi-agent systems model is studied. It has the discrete and unbounded distributed delays and is thus more in line with real systems. (2) The algorithm: two delayed impulsive pinning control schemes are proposed. The proposed controllers are more general than those in the literature, in which time delays in the impulsive pinning controller are not considered. If the impulses delays are equal to zero, the controller can reduce to a delay-free impulsive pinning controller. (3) The technique: based on Lyapunov stability theory, several novel mean square exponential consensus criteria are constructed. It is shown that the proposed delayed impulsive pinning controllers are effective.

The paper is organized as follows. In Sect. 2, the necessary mathematical preliminaries are presented. In Sect. 3, the control problem for stochastic multi-agent systems with unbounded distributed delays is formulated and certain criteria are presented. In Sect. 4, an illustrative example is presented to be demonstrated on the obtained conclusions. Section 5 gives the conclusion.

2 Model formulation and preliminaries

Stochastic multi-agent systems with N agents and unbounded distributed delays are considered. The dynamics of the ith ($i = 1, \ldots, N$) agent is described by

$$\mathrm{d}x_i(t) = \Big[-\mathcal{D}x_i(t) + \mathcal{A}\bar{f}(x_i(t)) + \mathcal{B}\bar{f}(x_i(t-\tau_1(t))) + c_1 \sum_{j=1}^{N} l_{ij}x_j(t)$$

$$+ c_2 \sum_{j=1}^{N} h_{ij} \int_{-\infty}^{t} K(t-s)\bar{f}(x_j(s))\mathrm{d}s$$
$$+ u_i(t)\Bigg]\mathrm{d}t + \bar{g}(x_i(t), x_i(t-\tau_2(t)))\mathrm{d}w(t) \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ denotes the state vector; $\mathcal{D}$, $\mathcal{A}$ and $\mathcal{B}$ are system matrices which are defined on $\mathbb{R}^{n\times n}$; c_1 and c_2 are coupling strengths; The discrete time-varying delays $0 \le \tau_1(t) \le \tau_1$ and $0 \le \tau_2(t) \le \tau_2$ satisfying $\max\{\tau_1, \tau_2\} = \tau$, in which τ is constant; $\bar{f}(x_i(t)) \in \mathbb{R}^n$ and $\bar{g}(x_i(t), x_i(t-\tau_2(t))) \in \mathbb{R}^n$ are continuous nonlinear vector functions; $w(t)$ is a scalar Brownian motion satisfying $\mathbb{E}\{[\mathrm{d}w(t)]\} = 0$ and $\mathbb{E}\{[\mathrm{d}w(t)]^2\} = \mathrm{d}t$; $u(t) = [u_1^T(t), \ldots, u_N^T(t)]^T$ is control input of agents; $L = (l_{ij})_{N\times N}$ and $H = (h_{ij})_{N\times N}$ are the undirected coupling matrix and distributed delay inner coupling matrices, respectively, which have the same definition, i.e.,

$$L = \begin{cases} l_{ij} \ge 0, & i \ne j, \\ l_{ii} = -\sum\limits_{j=1, j\ne i}^{N} l_{ij} & i = j. \end{cases}$$

$K(t)$ is a nonnegative bounded scalar function defined on $[0, +\infty)$, which represents the delay kernel of the unbounded distributed delay. The initial values are $x_i(t) = \phi_i(t), -\tau \le t \le 0$, where $\phi_i(t) \in \mathfrak{L}^2_{\mathbb{F}_0}([-\tau, 0], \mathbb{R}^n)$.

Remark 1 In model (1), the unbounded distributed time delays $\int_{-\infty}^{t} K(t-s)\bar{f}(x_j(s))\mathrm{d}s$ were involved. The delay variable s vary from $-\infty$ to t in a distributed way. Hence the unbounded distributed delays have a great influence on the consensus of the entire system, as can be seen in the conditions of Theorem. However, the existing methods are only valid for bounded distributed delays and there is very little existing work on stochastic multi-agent systems with unbounded distributed delays, and little work has been carried out on stochastic multi-agent systems with unbounded distributed delays. This is primarily due to the mathematical complications involved. Nevertheless, it is of considerable interesting to develop and explore methods for the consensus problem with unbounded distributed delays.

The leader is characterized by:

$$\mathrm{d}s(t) = [-\mathcal{D}s(t) + \mathcal{A}\bar{f}(s(t)) + \mathcal{B}\bar{f}(s(t-\tau_1(t)))]\mathrm{d}t$$
$$+ \bar{g}(s(t), s(t-\tau_2(t)))\mathrm{d}w(t) \tag{2}$$

where $s(t) \in \mathbb{R}^n$ denotes the leader state. The initial data are $s(t) = \psi(t) \in \mathfrak{L}^2_{\mathbb{F}_0}([-\tau, 0], \mathbb{R}^n)$. Denote $e_i(t) = x_i(t) - s(t)$. Constructing the impulsive pinning controller using the current and the previous error states for agent i.

$$u_i(t) = \begin{cases} \sum\limits_{k=1}^{+\infty}[\gamma_{1k}e_i(t) \\ \quad +\gamma_{2k}e_i(t-\tau_3(t))]\delta(t-t_k^-), & i \in \aleph_k \\ 0, & i \notin \aleph_k \end{cases} \tag{3}$$

where $\gamma_{1k}, \gamma_{2k} \in \mathbb{R}$ are the impulsive gains.$0 \le \tau_3(t) \le \tau_3$ is the impulse delay of controller. $\max\{\tau_1, \tau_2, \tau_3\} = \tau$. The Dirac function $\delta(\cdot)$ has the following property. $\int_{a-\varepsilon}^{a+\varepsilon} \delta(t)f(t-a)\mathrm{d}t = f(a)$ for $\varepsilon \ne 0$. $\aleph_k$ denotes the index set of pinned agents. We can reorder the error vector states, i.e., $\|e_{p_1}(t_k)\| \ge \|e_{p_2}(t_k)\| \ge \cdots \ge \|e_{p_{l_k}}(t_k)\| \ge \|e_{p_{l_{k+1}}}(t_k)\| \ge \cdots \ge \|e_{p_N}(t_k)\|$, then $\aleph_k = \{p_1, p_2, \ldots, p_{l_k}\}$ and $\sharp\aleph_k = l_k$, l_k denotes the number of agents pinned at each impulsive instant t_k.

Remark 2 As the number l_k of agents pinned at different impulsive instants is different, it is obvious that the proposed control scheme is more general than those in the literature, in which the number of agents pinned at different impulsive instants was assumed equal.

Remark 3 Recently, a great deal of research on impulsive pinning control in multi-agent systems has been conducted [21,22]. In these studies, time delays in the impulsive pinning controller were not considered. However, they should be taken into account owing to their extensive presence. In impulsive systems, they can be divided into two classes. One is internal time delays, which are due to limited transmission speed. The other is intrinsic time delays of impulsive controllers. Failure to consider these delays may lead to inaccurate or erroneous analysis results. In addition, communication delay is inevitable. Note that the time delays in the impulsive pinning controllers were neglected in [21,22]. Thus, it is vital to consider delays in impulsive pinning controllers.

Then the proposed impulsive pinning controller(3) is injected into the dynamical system. Due to $c_1 \sum\limits_{j=1}^{N} l_{ij}s(t)$

$= 0,\ c_2 \sum_{j=1}^{N} h_{ij} \int_{-\infty}^{t} K(t-u)\mathrm{f}(s(u))\mathrm{d}u = 0$, the error system can be described as

$$\begin{cases} de_i(t) &= \Big[-\mathcal{D}e_i(t) + \mathcal{A}\mathrm{f}(e_i(t)) \\ &\quad + \mathcal{B}\mathrm{f}(e_i(t-\tau_1(t))) + c_1 \sum_{j=1}^{N} l_{ij}e_j(t) \\ &\quad + c_2 \sum_{j=1}^{N} h_{ij} \int_{-\infty}^{t} K(t-s)\mathrm{f}(e_j(s))\mathrm{d}s \Big]\mathrm{d}t \\ &\quad + g(e_i(t), e_i(t-\tau_2(t)))\mathrm{d}w(t),\ t \neq t_k, \\ e_i(t_k) &= (1+\gamma_{1k})e_i(t_k^-) \\ &\quad + \gamma_{2k}e_i(t_k^- - \tau_3(t_k^-)),\ i \in \aleph_k \\ e_i(t_k) &= e_i(t_k^-),\ i \notin \aleph_k \\ e_i(t) &= \phi_i(t) - \psi(t) = \overline{\phi}_i(t), \quad -\tau \le t \le 0. \end{cases} \tag{4}$$

where $\overline{\phi}_i(t) \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. $\mathrm{f}(e_i(t)) = \bar{f}(x_i(t)) - \bar{f}(s(t))$, $g(e_i(t), e_i(t-\tau_2(t))) = \bar{g}(x_i(t), x_i(t-\tau_2(t))) - \bar{g}(s(t), s(t-\tau_2(t)))$. $x_i(t_k^+) = x_i(t_k)$ and $e_i(t_k^+ = e_i(t_k))$ imply that $x_i(t)$ and $e_i(t)$ are right continuous.

In the sequel, we impose the following assumptions on system.

(H_1) The function satisfies Lipschitz condition

$|\bar{f}_i(x_1) - \bar{f}_i(x_2)| \le \mu_i|x_1 - x_2|; \quad \forall x_1, x_2 \in \mathbb{R}^n.$

where $\mu_i (i = 1, 2, \ldots, n)$ is constant.

(H_2) For any $\bar{\mathrm{y}} > 0$, one has

$$\int_0^{+\infty} \mathrm{Y}(u)\mathrm{d}u = \bar{\mathrm{y}};$$

(H_3) For any vector $u(t)$, scalars $\rho_1 > 0$ and $\rho_2 > 0$, one has

$$\begin{aligned} &\mathrm{trace}[g^T(u(t), u(t-\tau(t)))g(u(t), u(t-\tau(t)))] \\ &\le \rho_1 u^T(t)u(t) + \rho_2 u^T(t-\tau(t))u(t-\tau(t)) \end{aligned}$$

Remark 4 Recently, researches on synchronization problems for network systems with bounded distributed delays have been conducted [12,14]. However, in these studies [12,14], the delay kernels were generally assumed to be one. It is obvious that the model considered in the present studies includes the cases in [12,14]. Moreover, the results obtained in this study are also applicable to dynamical systems with bounded delays. Thus our model is more general than those ones.

Throughout this paper, we will use these Lemmas in the sequel.

Lemma 1 ([28]). Let $a, b \in \mathbb{R}^n$, scalar $\epsilon > 0$, then one has

$$a^T b + b^T a \le \epsilon a^T a + \epsilon^{-1} b^T b. \tag{5}$$

Lemma 2 ([29])

$$\begin{aligned} &\lambda_{\min}(M^{-1}\Gamma)x^T M x \\ &\le x^T \Gamma x \le \lambda_{\max}(M^{-1}\Gamma)x^T M x, \forall x \in \mathbb{R}^n. \end{aligned} \tag{6}$$

where $M > 0 \in \mathbb{R}^{n\times n}$, $\Gamma \in \mathbb{R}^{n\times n}$ is symmetric.

Lemma 3 ([30]) Let $\varpi(t)$ satisfy the following impulsive differential inequality:

$$\begin{cases} D^+\varpi(t) & \le \rho\varpi(t) + \kappa_1[\varpi(t)]_{\tau_1} + \kappa_2[\varpi(t)]_{\tau_2} + \cdots \\ & \quad + \kappa_h[\varpi(t)]_{\tau_h}, t \neq t_k, t \ge t_0 \\ \varpi(t_k^+) & \le b_k\varpi(t_k^-) + d_k^1[\varpi(t_k^-)]_{\theta_1} + d_k^2[\varpi(t_k^-)]_{\theta_2} \\ & \quad + \cdots + d_k^r[\varpi(t_k^-)]_{\theta_r}, k \in \mathbb{N}^+ \\ \varpi(t) & = \phi(t), t \in [t_0 - \tau, t_0). \end{cases}$$

Suppose that

$$\begin{aligned} &b_k + \sum_{j=1}^{r} d_k^j < 1, \\ &\rho + \frac{\sum_{i=1}^{h} \kappa_i}{b_k + \sum_{j=1}^{r} d_k^j} + \frac{\ln(b_k + \sum_{j=1}^{r} d_k^j)}{(t_{k+1} - t_k)} < 0. \end{aligned}$$

Then for $\beta > 1$, $\lambda > 0$, one has

$$\varpi(t) \le \|\phi\|_\tau \beta e^{-\lambda(t-t_0)}, \quad t \ge t_0,$$

where $\|\phi\|_\tau = \sup_{t_0-\tau\le s\le t_0} \|\phi(s)\|$, $\tau = \max\{\tau_i, \theta_j, i = 1, 2, \ldots \bar{h},\ j = 1, 2, \ldots \bar{r}\}$, $\rho, \kappa_i, b_k, d_k^j,\ \tau_i, \theta_j$ are constants, and $\kappa_i \ge 0, b_k \ge 0, d_k^j \ge 0, \tau_i \ge 0, \theta_j \ge 0$, $[\varpi(t)]_{\tau_i} = \sup_{t-\tau_i\le s\le t} \varpi(s)$, $\left[\varpi(t_k^-)\right]_{\theta_j} = \sup_{t_k-\theta_j(t_k)\le s<t_k} \varpi(s)$, $\phi(t) \in \mathcal{C}([t_0-\tau, t_0], \mathbb{R}^+)$, and $\varpi(t)$ is continuous at $t \neq t_k, t \ge t_0$.

Lemma 4 ([23]) For $\alpha, \beta, \gamma \in \mathbb{R}$, then one has

$$(\alpha+\beta+\gamma)^2 \leq (1+\eta)\alpha^2 + (1+\eta^{-1})(1+\xi)\beta^2 + (1+\eta^{-1})(1+\xi^{-1})\gamma^2,$$

for any $\eta > 0, \xi > 0$.

3 Main results

Applying the impulsive pinning control, sufficient conditions are obtained whereby mean square exponential consensus of system (1) and (2) is ensured.

Theorem 1 *If assumption* $(H_1)-(H_3)$ *hold, for positive constants* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ *and a positive definite matrix* $P \leq \theta I_n$ *satisfying*

$$\mathfrak{b}_{1k} + \mathfrak{b}_{2k} < 1, \tag{7}$$

$$p + \frac{\sum_{i=1}^{3} q_i}{\mathfrak{b}_{1k} + \mathfrak{b}_{2k}} + \frac{ln(\mathfrak{b}_{1k} + \mathfrak{b}_{2k})}{t_{k+1} - t_k} < 0 \tag{8}$$

where $\mathfrak{b}'_{1k} = (1+\varepsilon_4)(1+\gamma_{1k})^2$, $\mathfrak{b}_{1k} = 1 - \frac{l_k}{N}(1-\mathfrak{b}'_{1k})$, $\mathfrak{b}_{2k} = (1+\varepsilon_4^{-1})\gamma_{2k}^2$, $\Pi = -2P\mathcal{D} + \varepsilon_1^{-1}P\mathcal{A}\mathcal{A}^T P + \varepsilon_1 \mu I_n + \varepsilon_2^{-1}P\mathcal{B}\mathcal{B}^T P + 2c_1\lambda_{\max}(L)P + c_2^2\lambda_{\max}^2(H)\varepsilon_3^{-1}P^2 + \theta\rho_1 I_n$, $p = \frac{\lambda_{\max}(\Pi)}{\lambda_{\min}(P)}$, $q_1 = \frac{\varepsilon_2\mu}{\lambda_{\min}(P)}$, $q_2 = \frac{\theta\rho_2}{\lambda_{\min}(P)}$, $q_3 = \frac{\varepsilon_3\bar{k}^2\mu}{\lambda_{\min}(P)}$, $\mu = \max\{\mu_i^2, i = 1, 2, \ldots, n\}$, *then the stochastic multi-agent system (4) with unbounded distributed delays can realize mean square exponential consensus.*

Proof Choose the following Lyapunov function

$$V(t) = \sum_{i=1}^{N} e_i^T(t)Pe_i(t).$$

According to Itô rule, we have

$$dV(t) = \mathscr{L}V(t)\mathrm{d}t + 2\sum_{i=1}^{N} e_i^T(t)Pg(e_i(t), e_i(t-\tau_2(t)))\mathrm{d}w(t),$$
$$t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+$$

where the operator $\mathscr{L}V(t)$ is

$$\begin{aligned}
&\mathscr{L}V(t) \\
&= 2\sum_{i=1}^{N} e_i^T(t)P\Big[-\mathcal{D}e_i(t) + \mathcal{A}\mathrm{f}(e_i(t)) \\
&\quad + \mathcal{B}\mathrm{f}(e_i(t-\tau_1(t))) + c_1\sum_{j=1}^{N} l_{ij}e_j(t) \\
&\quad + c_2\sum_{j=1}^{N} h_{ij}\int_{-\infty}^{t} K(t-s)\mathrm{f}(e_j(s))\mathrm{d}s\Big] \\
&\quad + \sum_{i=1}^{N} \mathrm{trace}\Big[g^T(e_i(t), e_i(t-\tau_2(t))) \\
&\quad \times Pg(e_i(t), e_i(t-\tau_2(t)))\Big] \\
&= -2\sum_{i=1}^{N} e_i^T(t)P\mathcal{D}e_i(t) + 2\sum_{i=1}^{N} e_i^T(t)P\mathcal{A}\mathrm{f}(e_i(t)) \\
&\quad + 2\sum_{i=1}^{N} e_i^T(t)P\mathcal{B}\mathrm{f}(e_i(t-\tau_1(t))) \\
&\quad + 2c_1\sum_{i=1}^{N}\sum_{j=1}^{N} l_{ij}e_i^T(t)Pe_j(t) \\
&\quad + 2c_2\sum_{i=1}^{N}\sum_{j=1}^{N} h_{ij}e_i^T(t)P\int_{-\infty}^{t} K(t-s)\mathrm{f}(e_j(s))\mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \mathrm{trace}\Big[g^T(e_i(t), e_i(t-\tau_2(t))) \\
&\quad \times Pg(e_i(t), e_i(t-\tau_2(t)))\Big]
\end{aligned} \tag{9}$$

Based on assumption $(H_1)-(H_3)$, one obtains

$$\begin{aligned}
&2e_i^T(t)P\mathcal{A}\mathrm{f}(e_i(t)) \\
&\leq \varepsilon_1^{-1}e_i^T(t)P\mathcal{A}\mathcal{A}^T Pe_i(t) + \varepsilon_1\mathrm{f}^T(e_i(t))\mathrm{f}(e_i(t)) \\
&\leq \varepsilon_1^{-1}e_i^T(t)P\mathcal{A}\mathcal{A}^T Pe_i(t) + \varepsilon_1\mu e_i^T(t)e_i(t)
\end{aligned} \tag{10}$$

$$\begin{aligned}
&2e_i^T(t)P\mathcal{B}\mathrm{f}(e_i(t-\tau_1(t))) \\
&\leq \varepsilon_2^{-1}e_i^T(t)P\mathcal{B}\mathcal{B}^T Pe_i(t) \\
&\quad + \varepsilon_2\mathrm{f}^T(e_i(t-\tau_1(t)))\mathrm{f}(e_i(t-\tau_1(t))) \\
&\leq \varepsilon_2^{-1}e_i^T(t)P\mathcal{B}\mathcal{B}^T Pe_i(t) \\
&\quad + \varepsilon_2\mu e_i^T(t-\tau_1(t))e_i(t-\tau_1(t))
\end{aligned} \tag{11}$$

where $\mu = \max\{\mu_i^2, i = 1, 2, \ldots, n\}$

$$\begin{aligned}
&2c_1\sum_{i=1}^{N}\sum_{j=1}^{N} l_{ij}e_i^T(t)Pe_j(t) \\
&= 2c_1\sum_{i=1}^{N}\sum_{j=1}^{N} l_{ij}\sum_{k=1}^{n} e_{ik}^T(t)Pe_{jk}(t) \\
&= 2c_1\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{n} l_{ij}e_{ik}^T(t)Pe_{jk}(t)
\end{aligned}$$

$$
\begin{aligned}
&= 2c_1 \sum_{k=1}^{n} (e^k(t))^T P L (e^k(t)) \\
&\le 2c_1 \lambda_{\max}(L) \sum_{i=1}^{N} e_i^T(t) P e_i(t)
\end{aligned} \tag{12}
$$

Using assumption (H_3), we obtain

$$
\begin{aligned}
&2c_2 \sum_{i=1}^{N} \sum_{j=1}^{N} h_{ij} e_i^T(t) P \int_{-\infty}^{t} K(t-s) \mathrm{f}(e_j(s)) \mathrm{d}s \\
&\le 2c_2 \lambda_{\max}(H) \sum_{i=1}^{N} e_i^T(t) P \int_{-\infty}^{t} K(t-s) \mathrm{f}(e_i(s)) \mathrm{d}s \\
&\le c_2^2 \lambda_{\max}^2(H) \varepsilon_3^{-1} \sum_{i=1}^{N} e_i^T(t) P^2 e_i(t) \\
&\quad + \varepsilon_3 \sum_{i=1}^{N} \left(\int_{-\infty}^{t} K(t-s) \mathrm{f}(e_i(s)) \mathrm{d}s \right)^T \\
&\quad \times \left(\int_{-\infty}^{t} K(t-s) \mathrm{f}(e_i(s)) \mathrm{d}s \right) \\
&\le c_2^2 \lambda_{\max}^2(H) \varepsilon_3^{-1} \sum_{i=1}^{N} e_i^T(t) P^2 e_i(t) \\
&\quad + \varepsilon_3 \bar{k} \sum_{i=1}^{N} \int_{-\infty}^{t} K(t-s) \mathrm{f}^T(e_i(s)) \mathrm{f}(e_i(s)) \mathrm{d}s \\
&\le c_2^2 \lambda_{\max}^2(H) \varepsilon_3^{-1} \sum_{i=1}^{N} e_i^T(t) P^2 e_i(t) \\
&\quad + \varepsilon_3 \bar{k} \mu \sum_{i=1}^{N} \int_{-\infty}^{t} K(t-s) e_i^T(s) e_i(s) \mathrm{d}s
\end{aligned} \tag{13}
$$

Note that the assumption $P \le \theta I_n$

$$
\begin{aligned}
&\sum_{i=1}^{N} \mathrm{trace} \Big[g^T(e_i(t), e_i(t-\tau_2(t))) \\
&\quad \times P g(e_i(t), e_i(t-\tau_2(t))) \Big] \\
&\le \sum_{i=1}^{N} \theta \Big(\rho_1 e_i^T(t) e_i(t) \\
&\quad + \rho_2 e_i^T(t-\tau_2(t)) e_i(t-\tau_2(t)) \Big)
\end{aligned} \tag{14}
$$

Substituting (10)–(14) into (9), one has

$$
\begin{aligned}
&\mathscr{L}V(t) \\
&\le \sum_{i=1}^{N} \Big(-2e_i^T(t) P \mathcal{D} e_i(t) + \varepsilon_1 \mu e_i^T(t) e_i(t) \\
&\quad + \varepsilon_1^{-1} e_i^T(t) P \mathcal{A} \mathcal{A}^T P e_i(t) + \varepsilon_2^{-1} e_i^T(t) P \mathcal{B} \mathcal{B}^T P e_i(t) \\
&\quad + \varepsilon_2 \mu e_i^T(t-\tau_1(t)) e_i(t-\tau_1(t)) \Big) \\
&\quad + 2c_1 \lambda_{\max}(L) \sum_{i=1}^{N} e_i^T(t) P e_i(t) \\
&\quad + c_2^2 \lambda_{\max}^2(H) \varepsilon_3^{-1} \sum_{i=1}^{N} e_i^T(t) P^2 e_i(t) \\
&\quad + \varepsilon_3 \bar{k} \mu \sum_{i=1}^{N} \int_{-\infty}^{t} K(t-s) e_i^T(s) e_i(s) \mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \theta \Big(\rho_1 e_i^T(t) e_i(t) + \rho_2 e_i^T(t-\tau_2(t)) e_i(t-\tau_2(t)) \Big) \\
&\le \sum_{i=1}^{N} e_i^T(t) \Big(-2P\mathcal{D} + \varepsilon_1^{-1} P \mathcal{A} \mathcal{A}^T P + \varepsilon_1 \mu I_n \\
&\quad + \varepsilon_2^{-1} P \mathcal{B} \mathcal{B}^T P + 2c_1 \lambda_{\max}(L) P \\
&\quad + c_2^2 \lambda_{\max}^2(H) \varepsilon_3^{-1} P^2 + \theta \rho_1 I_n \Big) e_i(t) \\
&\quad + \varepsilon_2 \mu \sum_{i=1}^{N} e_i^T(t-\tau_1(t)) e_i(t-\tau_1(t)) \\
&\quad + \theta \rho_2 \sum_{i=1}^{N} e_i^T(t-\tau_2(t)) e_i(t-\tau_2(t)) \Big) \\
&\quad + \varepsilon_3 \bar{k} \mu \sum_{i=1}^{N} \int_{-\infty}^{t} K(t-s) e_i^T(s) e_i(s) \mathrm{d}s
\end{aligned} \tag{15}
$$

Let $\Pi = -2P\mathcal{D} + \varepsilon_1^{-1} P \mathcal{A} \mathcal{A}^T P + \varepsilon_1 \mu I_n + \varepsilon_2^{-1} P \mathcal{B} \mathcal{B}^T P + 2c_1 \lambda_{\max}(L) P + c_2^2 \lambda_{\max}^2(H) \varepsilon_3^{-1} P^2 + \theta \rho_1 I_n$, $p = \frac{\lambda_{\max}(\Pi)}{\lambda_{\min}(P)}$, $q_1 = \frac{\varepsilon_2 \mu}{\lambda_{\min}(P)}$, $q_2 = \frac{\theta \rho_2}{\lambda_{\min}(P)}$, $q_3 = \frac{\varepsilon_3 \bar{k}^{-2} \mu}{\lambda_{\min}(P)}$, then

$$
\begin{aligned}
\mathscr{L}V(t) &\le pV(t) + q_1 V(t-\tau_1(t)) + q_2 V(t-\tau_2(t)) \\
&\quad + \varepsilon_3 \bar{k} \mu \sum_{i=1}^{N} \int_{-\infty}^{t} K(t-s) e_i^T(s) e_i(s) \mathrm{d}s
\end{aligned} \tag{16}
$$

Taking mathematical expectations of Eq. (16), it yields

$$D^{+}\mathbb{E}\{V(t)\} = \mathbb{E}\{\mathscr{L}V(t)\}$$
$$\leq p\mathbb{E}\{V(t)\} + q_1[\mathbb{E}\{V(t)\}]_{\tau_1} + q_2[\mathbb{E}\{V(t)\}]_{\tau_2}$$
$$+ \frac{\varepsilon_3 \bar{k}^{-2}\mu}{\lambda_{\min}(P)}[\mathbb{E}\{V(t)\}]_{-\infty}$$
$$= p\mathbb{E}\{V(t)\} + q_1[\mathbb{E}\{V(t)\}]_{\tau_1} + q_2[\mathbb{E}\{V(t)\}]_{\tau_2}$$
$$+ q_3[\mathbb{E}\{V(t)\}]_{-\infty}, \quad t \in [t_{k-1}, t_k), \tag{17}$$

where $[\mathbb{E}\{V(t)\}]_{-\infty} = \max_{s\leq t}\mathbb{E}\{V(t)\}$.

When $t = t_k$, according to (4),

$$V(t_k^+) = \sum_{k=1}^{N} e_i^T(t_k^+)Pe_i(t_k^+)$$
$$= \sum_{i\in\aleph(t_k)} e_i^T(t_k^+)Pe_i(t_k^+) + \sum_{i\notin\aleph(t_k)} e_i^T(t_k^+)Pe_i(t_k^+)$$
$$= \sum_{i\in\aleph(t_k)} \Big((1+\gamma_{1k})e_i(t_k^-) + \gamma_{2k}e_i(t_k^- - \tau_3(t_k^-))\Big)^T$$
$$\times P\Big((1+\gamma_{1k})e_i(t_k^-) + \gamma_{2k}e_i(t_k^- - \tau_3(t_k^-))\Big)$$
$$+ \sum_{i\notin\aleph(t_k)} e_i^T(t_k^-)Pe_i(t_k^-)$$
$$\leq \sum_{i\in\aleph(t_k)} [(1+\gamma_{1k})^2 e_i^T(t_k^-)Pe_i(t_k^-)$$
$$+ \gamma_{2k}^2 e_i^T(t_k^- - \tau_3(t_k^-))Pe_i(t_k^- - \tau_3(t_k^-))$$
$$+ 2(1+\gamma_{1k})\gamma_{2k}e_i^T(t_k^-)Pe_i(t_k^- - \tau_3(t_k^-))]$$
$$+ \sum_{i\notin\aleph(t_k)} e_i^T(t_k^-)Pe_i(t_k^-)$$
$$\leq (1+\varepsilon_4)(1+\gamma_{1k})^2 \sum_{i\in\aleph(t_k)} e_i^T(t_k^-)Pe_i(t_k^-)$$
$$+ (1+\varepsilon_4^{-1})\gamma_{2k}^2 \sum_{i\in\aleph(t_k)} e_i^T(t_k^- - \tau_3(t_k^-))Pe_i(t_k^- - \tau_3(t_k^-))$$
$$+ \sum_{i\notin\aleph(t_k)} e_i^T(t_k^-)Pe_i(t_k^-) \tag{18}$$

In view of pinning control, we have

$$\sum_{i\notin\aleph(t_k)} e_i^T(t_k^+)Pe_i(t_k^+) \leq \frac{N-l_k}{N}\sum_{i=1}^{N} e_i^T(t_k^-)Pe_i(t_k^-) \tag{19}$$

$$\sum_{i\in\aleph(t_k)} e_i^T(t_k^-)Pe_i(t_k^-) \leq \frac{l_k}{N}\sum_{i=1}^{N} e_i^T(t_k^-)Pe_i(t_k^-) \tag{20}$$

Substituting (19) and (20) into (18), then we have

$$V(t_k^+) \leq \mathfrak{b}'_{1k}\frac{l_k}{N}\sum_{i=1}^{N} e_i^T(t_k^-)Pe_i(t_k^-)$$
$$+ \frac{N-l_k}{N}\sum_{i=1}^{N} e_i^T(t_k^-)Pe_i(t_k^-)$$
$$+ \mathfrak{b}_{2k}\sum_{i\in\aleph(t_k)} e_i^T(t_k^- - \tau_3(t_k^-))Pe_i(t_k^- - \tau_3(t_k^-))$$
$$= \mathfrak{b}_{1k}V(t_k^-) + \mathfrak{b}_{2k}[V(t_k^-)]_{\tau_3} \tag{21}$$

where $\mathfrak{b}'_{1k} = (1+\varepsilon_4)(1+\gamma_{1k})^2$, $\mathfrak{b}_{1k} = 1 - \frac{l_k}{N}(1-\mathfrak{b}'_{1k})$, $\mathfrak{b}_{2k} = (1+\varepsilon_4^{-1})\gamma_{2k}^2$.

Then it yields

$$\mathbb{E}\{V(t_k^+)\} \leq \mathfrak{b}_{1k}\mathbb{E}V(t_k^-) + \mathfrak{b}_{2k}\mathbb{E}[V(t_k^-)]_{\tau_3} \tag{22}$$

In view of Lemma 3, if the condition (7) and (8) hold, we can deduce from (17) and (20) that

$$\mathbb{E}\{V(t)\} \leq \chi e^{-\eta t}, \ t \geq 0. \tag{23}$$

where $\chi = \lambda_{\max}(P)M\mathbb{E}\sum_{i=1}^{N}\sup_{-\tau\leq s\leq 0}\{\|\overline{\phi}_i(s)\|^2\}$, $\eta > 0$, $M > 1$. The systems (1) and (2) can be reached mean square exponential consensus with the impulsive pinning control. The proof is completed. □

Remark 5 The result is not related to discrete time delay, which is in agreement with the general result. It should be noted that Theorem 1 is related to $\bar{k}$, which implies that distributed delays play an important role. As most existing results concern the delay-free impulsive control problem or the bounded distributed delay problem. The results in this study are new and more general.

Remark 6 In order to minimize the term $\mathfrak{b}_{1k} + \mathfrak{b}_{2k}$, we define the function as $h_k(\varepsilon_4) = 1 - \frac{l_k}{N}(1-(1+\varepsilon_4)(1+\gamma_{1k})^2) + (1+\varepsilon_4^{-1})\gamma_{2k}^2$. Then let $h'_k(\varepsilon_4) = 0$, we obtain $\varepsilon_4 = \frac{|\gamma_{2k}|}{|1+\gamma_{1k}|}\sqrt{\frac{N}{l_k}}$. Hence, $\min\{\mathfrak{b}_{1k}+\mathfrak{b}_{2k}\} = \min\{h_k(\varepsilon_4)\} = 1 - \frac{l_k}{N} + [\sqrt{\frac{l_k}{N}}|1+\gamma_{1k}| + |\gamma_{2k}|]^2$. As l_k represents the number of agents controlled at impulsive instant $t = t_k$. $\frac{l_k}{N}$ is called pinning ratio and the proportion of impulsive controlled agents at $t = t_k$. The pinning ratio varies with impulse time. In addition, it is related to impulsive gain.

If $\gamma_{1k}=0$, then the impulsive pinning control protocol can be written as $\min\{\mathfrak{b}_{1k}+\mathfrak{b}_{2k}\}=1+2\sqrt{\frac{l_k}{N}}|\gamma_{2k}|+|\gamma_{2k}|^2>1$. Hence, condition (7) in Theorem 1 is not satisfied; thus, Theorem 1 cannot be used to analyze this case according to this estimation method. In the next section, the estimation method will be changed, and the following delayed impulsive pinning controller will be discussed.

$$u_i(t)=\begin{cases}\sum_{k=1}^{+\infty}\gamma_k e_i(t-\tau_3(t))\delta(t-t_k^-), & i\in\aleph_k\\ 0, & i\notin\aleph_k\end{cases}\tag{24}$$

Then, the error system is described as

$$\begin{cases}\mathrm{d}e_i(t)=\Big[-\mathcal{D}e_i(t)+\mathcal{A}\mathrm{f}(e_i(t))\\ \quad+\mathcal{B}\mathrm{f}(e_i(t-\tau_1(t)))+c_1\sum_{j=1}^{N}l_{ij}e_j(t)\\ \quad+c_2\sum_{j=1}^{N}h_{ij}\int_{-\infty}^{t}K(t-s)\mathrm{f}(e_j(s))\mathrm{d}s\Big]\mathrm{d}t\\ \quad+g(e_i(t),e_i(t-\tau_2(t)))\mathrm{d}w(t),\ t\neq t_k,\\ e_i(t_k)=e_i(t_k^-)+\gamma_k e_i(t_k^- -\tau_3(t_k^-)),\ i\in\aleph_k\\ e_i(t_k)=e_i(t_k^-),\ i\notin\aleph_k\\ e_i(t)=\phi_i(t)-\psi(t)=\overline{\phi}_i(t),\quad -\tau\le t\le 0.\end{cases}\tag{25}$$

Theorem 2 *If the assumption $(H_1)-(H_3)$ hold, for positive constants $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ and positive definite matrix $P\le\theta I_n$ satisfying*

$$\sum_{j=1}^{5}\mathfrak{b}_{jk}<1,\tag{26}$$

$$p+\frac{\sum_{i=1}^{3}q_i}{\sum_{j=1}^{5}\mathfrak{b}_{jk}}+\frac{\ln(\sum_{j=1}^{5}\mathfrak{b}_{jk})}{t_{k+1}-t_k}<0\tag{27}$$

where

$$\mathfrak{b}'_{1k}=\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\Big((1+\varepsilon_1)(1+\gamma_k)^2\frac{l_k}{N}+(1+\varepsilon_1^{-1})(1+\xi_1)\Xi\Big),$$

$$\mathfrak{b}_{1k}=1-\frac{l_k}{N}+\mathfrak{b}'_1,$$

$$\mathfrak{b}_{2k}=\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(1+\varepsilon_1^{-1})(1+\xi_1)\Xi_1,$$

$$\mathfrak{b}_{3k}=\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(1+\varepsilon_1^{-1})(1+\xi_1)\Xi_2,$$

$$\mathfrak{b}_{4k}=\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(1+\varepsilon_1^{-1})(1+\xi_1)\Xi_3,$$

$$\mathfrak{b}_{5k}=\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(1+\varepsilon_1^{-1})(1+\xi_1^{-1})\gamma_k^4\varsigma^2\frac{l_k}{N},$$

$$\begin{aligned}\Xi&=2\gamma_k^2\tau_3^2(1+\varepsilon_2)(1+\varepsilon_3)\lambda_{\max}(\mathcal{D}^2)\\&\quad+2\gamma_k^2\tau_3^2(1+\varepsilon_2)(1+\varepsilon_3^{-1})(1+\xi_3)\mu\lambda_{\max}(\mathcal{A}^2),\\&\quad+2\gamma_k^2\tau_3^2Nl_kc_1(1+\varepsilon_2^{-1})(1+\xi_2)\max_{i,j}\{l_{ij}^2\}\\&\quad+2\gamma_k^2\tau_3\frac{l_k}{N}\rho_1,\end{aligned}$$

$$\Xi_1=2\gamma_k^2\tau_3^2(1+\varepsilon_2)(1+\varepsilon_3^{-1})(1+\xi_3^{-1})\mu\lambda_{\max}(\mathcal{B}^2),$$

$$\Xi_2=2\gamma_k^2\tau_3\frac{l_k}{N}\rho_2,$$

$$\Xi_3=2\gamma_k^2\tau_3^2Nl_kc_2(1+\varepsilon_2^{-1})(1+\xi_2^{-1})\bar{k}^2\mu\max_{i,j}\{h_{ij}^2\},$$

$$\begin{aligned}\Pi&=-2P\mathcal{D}+\varepsilon_1^{-1}P\mathcal{A}\mathcal{A}^TP+\varepsilon_1\mu I_n+\varepsilon_2^{-1}P\mathcal{B}\mathcal{B}^TP\\&\quad+2c_1\lambda_{\max}(L)P+c_2^2\lambda_{\max}^2(H)\varepsilon_3^{-1}P^2+\theta\rho_1I_n,\end{aligned}$$

$$p=\frac{\lambda_{\max}(\Pi)}{\lambda_{\min}(P)},\ q_1=\frac{\varepsilon_2\mu}{\lambda_{\min}(P)},\ q_2=\frac{\theta\rho_2}{\lambda_{\min}(P)},$$

$$q_3=\frac{\varepsilon_3\bar{k}^2\mu}{\lambda_{\min}(P)},\ \mu=\max\{\mu_i^2,i=1,2,\ldots,n\},$$

then the system (25) with unbounded distributed delays can reach mean square exponential consensus.

Proof Construct the similar Lyapunov function

$$V(t)=\sum_{i=1}^{N}e_i^T(t)Pe_i(t).$$

When $t\in[t_{k-1},t_k),k\in\mathbb{N}^+$, one can obtain the same estimation of inequality. Let $\Pi=-2P\mathcal{D}+\varepsilon_1^{-1}P\mathcal{A}\mathcal{A}^TP+\varepsilon_1\mu I_n+\varepsilon_2^{-1}PBB^TP+2c_1\lambda_{\max}(L)P+c_2^2\lambda_{\max}^2(H)\varepsilon_3^{-1}P^2+\theta\rho_1I_n$, $p=\frac{\lambda_{\max}(\Pi)}{\lambda_{\min}(P)}$, $q_1=\frac{\varepsilon_2\mu k}{\lambda_{\min}(P)}$, $q_2=\frac{\theta\rho_2 k}{\lambda_{\min}(P)}$, $q_3=\frac{\varepsilon_3\bar{k}^2\mu}{\lambda_{\min}(P)}$,

$$\begin{aligned}\mathscr{L}V(t)&\le pV(t)+q_1V(t-\tau_1(t))+q_2V(t-\tau_2(t))\\&\quad+\varepsilon_3\bar{k}\mu\sum_{i=1}^{N}\int_{-\infty}^{t}K(t-s)e_i^T(s)e_i(s)\mathrm{d}s\end{aligned}\tag{28}$$

It yields

$$D^+\mathbb{E}\{V(t)\}=\mathbb{E}\{\mathscr{L}V(t)\}$$

$$
\begin{aligned}
&\leq p\mathbb{E}\{V(t)\}+q_1[\mathbb{E}\{V(t)\}]_{\tau_1}+q_2[\mathbb{E}\{V(t)\}]_{\tau_2}\\
&\quad+\frac{\varepsilon_3\overline{k}^2\mu}{\lambda_{\min}(P)}[\mathbb{E}\{V(t)\}]_{-\infty}\\
&=p\mathbb{E}\{V(t)\}+q_1[\mathbb{E}\{V(t)\}]_{\tau_1}+q_2[\mathbb{E}\{V(t)\}]_{\tau_2}\\
&\quad+q_3[\mathbb{E}\{V(t)\}]_{-\infty},\quad t\in(t_{k-1},t_k], \qquad (29)
\end{aligned}
$$

When $t=t_k$, both sides of the first term of (25) take integral, then we have

$$
\begin{aligned}
&e_i(t_k^-)-e_i(t_k^- -\tau_3(t_k^-))\\
&=\int_{t_k^- -\tau_3(t_k^-)}^{t_k^-}\Big[-\mathcal{D}e_i(\nu)+\mathcal{A}\mathrm{f}(e_i(\nu))\\
&\quad+\mathcal{B}f(e_i(\nu-\tau_1(\nu)))+c_1\sum_{j=1}^{N}l_{ij}e_j(\nu)\\
&\quad+c_2\sum_{j=1}^{N}h_{ij}\int_{-\infty}^{t}K(\nu-s)\mathrm{f}(e_j(s))\mathrm{d}s\Big]\mathrm{d}\nu\\
&\quad+\int_{t_k^- -\tau_3(t_k^-)}^{t_k^-}g(e_i(\nu),e_i(\nu-\tau_2(\nu)))\mathrm{d}w(\nu)\\
&\quad+\sum_{m=1}^{\zeta_k}\gamma_k e_i(t_{k-m}^- -\tau_3(t_k^-)) \qquad (30)
\end{aligned}
$$

where ς_k represents the amount of impulses injecting into agent i during the period $(t_k^- - \tau_3(t_k^-), t_k^-)$. Because some impulses may be injected into other agents according to the pinning scheme, then $\varsigma_k \leq \varsigma$, $\delta = t_k - t_{k-1}$. ς denotes the number of impulses that the networks subject to on time span $(t_k^- - \tau_3(t_k^-), t_k^-)$, for $k \in \mathbb{N}^+$, which is described as

$$
\varsigma=\begin{cases}\lfloor\frac{d}{\delta}\rfloor & \text{if } mod(d,\delta)\neq 0,\\ \lfloor\frac{d}{\delta}\rfloor-1 & \text{if } mod(d,\delta)=0.\end{cases}
$$

For $i \in \aleph_k$, $t = t_k^-$, according to (30), we have

$$
\begin{aligned}
&e_i(t_k)=e_i(t_k^-)+\gamma_k e_i(t_k^-)\\
&\quad-\gamma_k\int_{t_k^- -\tau_3(t_k^-)}^{t_k^-}\Big[-\mathcal{D}e_i(\nu)+\mathcal{A}\mathrm{f}(e_i(\nu))\\
&\quad+\mathcal{B}\mathrm{f}(e_i(\nu-\tau_1(\nu)))+c_1\sum_{j=1}^{N}l_{ij}e_j(\nu)\\
&\quad+c_2\sum_{j=1}^{N}h_{ij}\int_{-\infty}^{t}K(\nu-s)\mathrm{f}(e_j(s))\mathrm{d}s\Big]\mathrm{d}\nu\\
&\quad-\gamma_k\int_{t_k^- -\tau_3(t_k^-)}^{t_k^-}g(e_i(\nu),e_i(\nu-\tau_2(\nu)))\mathrm{d}w(\nu)\\
&\quad-\gamma_k^2\sum_{m=1}^{\varsigma_k}e_i(t_{k-m}^- -\tau_3(t_k^-)) \qquad (31)
\end{aligned}
$$

Let

$$
\begin{aligned}
Y_{i1}&=(1+\gamma_k)e_i(t_k^-)\\
Y_{i2}&=-\gamma_k\int_{t_k^- -\tau_3(t_k^-)}^{t_k^-}\Big[-\mathcal{D}e_i(\nu)\\
&\quad+\mathcal{A}\mathrm{f}(e_i(\nu))+\mathcal{B}\mathrm{f}(e_i(\nu-\tau_1(\nu)))+c_1\sum_{j=1}^{N}l_{ij}e_j(\nu)\\
&\quad+c_2\sum_{j=1}^{N}h_{ij}\int_{-\infty}^{t}K(\nu-s)\mathrm{f}(e_j(s))\mathrm{d}s\Big]\mathrm{d}\nu\\
&\quad-\gamma_k\int_{t_k^- -\tau_3(t_k^-)}^{t_k^-}g(e_i(\nu),e_i(\nu-\tau_2(\nu)))\mathrm{d}w(\nu)\\
Y_{i3}&=-\gamma_k^2\sum_{m=1}^{\varsigma_k}e_i(t_{k-m}^- -\tau_3(t_k^-))
\end{aligned}
$$

Then applying Lemma 4, we have

$$
\begin{aligned}
&\sum_{i\in\aleph(t_k)}e_i^2(t_k^+)=\sum_{i\in\aleph(t_k)}\{Y_{i1}+Y_{i2}+Y_{i3}\}^2\\
&\leq(1+\varepsilon_1)\sum_{i\in\aleph(t_k)}Y_{i1}^2\\
&\quad+(1+\varepsilon_1^{-1})(1+\xi_1)\sum_{i\in\aleph(t_k)}Y_{i2}^2\\
&\quad+(1+\varepsilon_1^{-1})(1+\xi_1^{-1})\sum_{i\in\aleph(t_k)}Y_{i3}^2 \qquad (32)
\end{aligned}
$$

Then calculating item by item and applying Lemma 4 and Schwarz's inequality, it yields

$$
\begin{aligned}
&(1+\varepsilon_1)\sum_{i\in\aleph(t_k)}Y_{i1}^2\\
&=(1+\varepsilon_1)(1+\gamma_k)^2\sum_{i\in\aleph(t_k)}e_i^2(t_k^-)\\
&\leq\left((1+\varepsilon_1)(1+\gamma_k)^2\frac{l_k}{N}\right)\frac{1}{\lambda_{\min}(P)}V(t) \qquad (33)
\end{aligned}
$$

On account of $(a+b)^2 \le 2a^2 + 2b^2$, we obtain that

$$\sum_{i\in\aleph(t_k)} Y_{i2}^2$$
$$\le 2\gamma_k^2 \sum_{i\in\aleph(t_k)} \Bigg\{ \int_{t_k^- - \tau_3(t_k^-)}^{t_k^-} \Big[-\mathcal{D}e_i(\nu) + \mathcal{A}\mathrm{f}(e_i(\nu))$$
$$+\mathcal{B}\mathrm{f}(e_i(\nu-\tau_1(\nu))) + c_1 \sum_{j=1}^N l_{ij} e_j(\nu)$$
$$+c_2 \sum_{j=1}^N h_{ij} \int_{-\infty}^t K(\nu - s)\mathrm{f}(e_j(s))\mathrm{d}s \Big] \mathrm{d}\nu \Bigg\}^2$$
$$+2\gamma_k^2 \sum_{i\in\aleph(t_k)} \Bigg\{ \int_{t_k^- - \tau_3(t_k^-)}^{t_k^-} g(e_i(\nu), e_i(\nu - \tau_2(\nu)))\mathrm{d}w(\nu) \Bigg\}^2$$
$$\le 2\gamma_k^2 \tau_3 \sum_{i\in\aleph(t_k)} \int_{t_k^- - \tau_3(t_k^-)}^{t_k^-} \Big[-\mathcal{D}e_i(\nu) + \mathcal{A}\mathrm{f}(e_i(\nu))$$
$$+\mathcal{B}\mathrm{f}(e_i(\nu-\tau_1(\nu))) + c_1 \sum_{j=1}^N l_{ij} e_j(\nu)$$
$$+c_2 \sum_{j=1}^N h_{ij} \int_{-\infty}^t K(\nu - s)\mathrm{f}(e_j(s))\mathrm{d}s \Big]^2 \mathrm{d}\nu$$
$$+2\gamma_k^2 \sum_{i\in\aleph(t_k)} \int_{t_k^- - \tau_3(t_k^-)}^{t_k^-} [g(e_i(\nu), e_i(\nu - \tau_2(\nu)))]^2 \mathrm{d}\nu$$
$$\le 2\gamma_k^2 \tau_3 \int_{t_k^- - \tau_3(t_k^-)}^{t_k^-} (1+\varepsilon_2) \sum_{i\in\aleph(t_k)}$$
$$\times \Big[-\mathcal{D}e_i(\nu) + \mathcal{A}\mathrm{f}(e_i(\nu)) + \mathcal{B}\mathrm{f}(e_i(\nu-\tau_1(\nu))) \Big]^2$$
$$+c_1(1+\varepsilon_2^{-1})(1+\xi_2) \sum_{i\in\aleph(t_k)} \Bigg(\sum_{j=1}^N l_{ij} e_j(\nu) \Bigg)^2$$
$$+c_2(1+\varepsilon_2^{-1})(1+\xi_2^{-1})$$
$$\times \sum_{i\in\aleph(t_k)} \Bigg[\sum_{j=1}^N h_{ij} \int_{-\infty}^t K(\nu - s)\mathrm{f}(e_j(s))\mathrm{d}s \Bigg]^2 \mathrm{d}\nu$$
$$+2\gamma_k^2 \sum_{i\in\aleph(t_k)} \int_{t_k^- - \tau_3(t_k^-)}^{t_k^-} (\rho_1 \|e_i(\nu)\|^2$$
$$+\rho_2 \|e_i(\nu - \tau_2(\nu))\|^2)\mathrm{d}\nu$$
$$\le 2\gamma_k^2 \tau_3 \int_{t_k^- - \tau_3(t_k^-)}^{t_k^-} (1+\varepsilon_2) \Bigg\{ (1+\varepsilon_3)\lambda_{\max}(\mathcal{D}^2)$$
$$\times \sum_{i\in\aleph(t_k)} e_i^2(\nu)$$
$$+(1+\varepsilon_3^{-1})(1+\xi_3)\lambda_{\max}(\mathcal{A}^2) \sum_{i\in\aleph(t_k)} \mathrm{f}^2(e_i(\nu))$$
$$+(1+\varepsilon_3^{-1})(1+\xi_3^{-1})\lambda_{\max}(\mathcal{B}^2)$$
$$\times \sum_{i\in\aleph(t_k)} \mathrm{f}^2(e_i(\nu - \tau_1(\nu))) \Bigg\}$$
$$+Nc_1(1+\varepsilon_2^{-1})(1+\xi_2) \sum_{i\in\aleph(t_k)} \sum_{j=1}^N l_{ij}^2 e_j^2(\nu)$$
$$+Nc_2(1+\varepsilon_2^{-1})(1+\xi_2^{-1})\bar{k}\mu$$
$$\times \sum_{i\in\aleph(t_k)} \Bigg[\sum_{j=1}^N h_{ij}^2 \int_{-\infty}^t K(\nu - s)e_j^2(s)\mathrm{d}s \Bigg] \mathrm{d}\nu$$
$$+2\gamma_k^2 \sum_{i\in\aleph(t_k)} \int_{t_k^- - \tau_3(t_k^-)}^{t_k^-} (\rho_1 \|e_i(\nu)\|^2$$
$$+\rho_2 \|e_i(\nu - \tau_2(\nu))\|^2)\mathrm{d}\nu$$
$$\le 2\gamma_k^2 \tau_3^2 (1+\varepsilon_2)(1+\varepsilon_3)\lambda_{\max}(\mathcal{D}^2) \sum_{i=1}^N e_i^2(\nu)$$
$$+2\gamma_k^2 \tau_3^2 (1+\varepsilon_2)(1+\varepsilon_3^{-1})$$
$$\times (1+\xi_3)\mu\lambda_{\max}(\mathcal{A}^2) \sum_{i=1}^N e_i^2(\nu)$$
$$+2\gamma_k^2 \tau_3^2 (1+\varepsilon_2)(1+\varepsilon_3^{-1})$$
$$\times (1+\xi_3^{-1})\mu\lambda_{\max}(\mathcal{B}^2) \sum_{i=1}^N e_i^2(\nu - \tau_1(\nu))$$
$$+2\gamma_k^2 \tau_3^2 N l_k c_1 (1+\varepsilon_2^{-1})(1+\xi_2) \max_{i,j}\{l_{ij}^2\} \sum_{i=1}^N e_i^2(\nu)$$
$$+2\gamma_k^2 \tau_3^2 N l_k c_2 (1+\varepsilon_2^{-1})(1+\xi_2^{-1})\bar{k}\mu$$
$$\times \max_{i,j}\{h_{ij}^2\} \sum_{j=1}^N \int_{-\infty}^t K(\nu - s)e_j^2(s)\mathrm{d}s$$
$$+2\gamma_k^2 \tau_3 \frac{l_k}{N} (\rho_1 \sum_{i=1}^N \|e_i(\nu)\|^2$$
$$+\rho_2 \sum_{i=1}^N \|e_i(\nu - \tau_2(\nu))\|^2)$$
$$\le \frac{1}{\lambda_{\min}(P)} \Big(\Xi V(t_k^-) + \Xi_1 [V(t_k^-)]_{\tau_1}$$
$$+\Xi_2 [V(t_k^-)]_{\tau_2} + \Xi_3 [V(t_k^-)]_{-\infty} \Big) \tag{34}$$

where

$$\begin{aligned}\Xi &= 2\gamma_k^2\tau_3^2(1+\varepsilon_2)(1+\varepsilon_3)\lambda_{\max}(\mathcal{D}^2)\\&\quad+2\gamma_k^2\tau_3^2(1+\varepsilon_2)(1+\varepsilon_3^{-1})(1+\xi_3)\mu\lambda_{\max}(\mathcal{A}^2)\\&\quad+2\gamma_k^2\tau_3^2Nl_kc_1(1+\varepsilon_2^{-1})(1+\xi_2)\max_{i,j}\{l_{ij}^2\}\\&\quad+2\gamma_k^2\tau_3\frac{l_k}{N}\rho_1\\\Xi_1 &= 2\gamma_k^2\tau_3^2(1+\varepsilon_2)(1+\varepsilon_3^{-1})(1+\xi_3^{-1})\mu\lambda_{\max}(\mathcal{B}^2)\\\Xi_2 &= \gamma_k^2\tau_3\frac{l_k}{N}\rho_2\\\Xi_3 &= 2\gamma_k^2\tau_3^2Nl_kc_2(1+\varepsilon_2^{-1})(1+\xi_2^{-1})\bar{k}^2\mu\max_{i,j}\{h_{ij}^2\}\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in\aleph(t_k)}Y_{i3}^2 &\le \gamma_k^4\sum_{i\in\aleph(t_k)}\left(\sum_{m=1}^{\varsigma_k}e_i(t_{k-m}^- - \tau_3(t_k^-))\right)^2\\&\le \gamma_k^4\varsigma_k\sum_{i\in\aleph(t_k)}\sum_{m=1}^{\varsigma_k}e_i^2(t_{k-m}^- - \tau_3(t_k^-))\\&\le \gamma_k^4\varsigma\frac{l_k}{N}\sum_{m=1}^{\varsigma_k}\sum_{i=1}^{N}e_i^2(t_{k-m}^- - \tau_3(t_k^-))\\&\le \frac{1}{\lambda_{\min}(P)}\gamma_k^4\varsigma^2\frac{l_k}{N}[V(t_k^-)]_{\tau_3} \qquad (35)\end{aligned}$$

Substituting (33)–(35) into (32), we have

$$\begin{aligned}&\sum_{i\in\aleph(t_k)}e_i^2(t_k^+)\\&\le (1+\varepsilon_1)(1+\gamma_k)^2\frac{l_k}{N}\frac{1}{\lambda_{\min}(P)}V(t_k^-)\\&\quad+(1+\varepsilon_1^{-1})(1+\xi_1)\frac{1}{\lambda_{\min}(P)}\Big(\Xi V(t)\\&\quad+\Xi_1[V(t_k^-)]_{\tau_1}+\Xi_2[V(t_k^-]_{\tau_2}+\Xi_3[V(t_k^-]_{-\infty}\Big)\\&\quad+(1+\varepsilon_1^{-1})(1+\xi_1^{-1})\frac{1}{\lambda_{\min}(P)}\left(\gamma_k^4\varsigma^2\frac{l_k}{N}[V(t_k^-)]_{\tau_3}\right) \qquad (36)\end{aligned}$$

$$\begin{aligned}V(t_k^+) &= \sum_{k=1}^{N}e_i^T(t_k^+)Pe_i(t_k^+)\\&= \sum_{i\in\aleph(t_k)}e_i^T(t_k^+)Pe_i(t_k^+)+\sum_{i\notin\aleph(t_k)}e_i^T(t_k^+)Pe_i(t_k^+)\\&\le \lambda_{\max}(P)\sum_{i\in\aleph(t_k)}e_i^T(t_k^+)e_i(t_k^+)\\&\quad+\sum_{i\notin\aleph(t_k)}e_i^T(t_k^+)Pe_i(t_k^+)\\&\le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\Bigg\{\bigg((1+\varepsilon_1)(1+\gamma_k)^2\frac{l_k}{N}\\&\quad+(1+\varepsilon_1^{-1})(1+\xi_1)\Xi\bigg)V(t_k^-)\\&\quad+(1+\varepsilon_1^{-1})(1+\xi_1)\Xi_1[V(t_k^-)]_{\tau_1}\\&\quad+(1+\varepsilon_1^{-1})(1+\xi_1)\Xi_2[V(t_k^-)]_{\tau_2}\\&\quad+(1+\varepsilon_1^{-1})(1+\xi_1)\Xi_3[V(t_k^-)]_{-\infty}\\&\quad+(1+\varepsilon_1^{-1})(1+\xi_1^{-1})\gamma_k^4\varsigma^2\frac{l}{N}[V(t_k^-)]_{\tau_3}\Bigg\}\\&\quad+\sum_{i\notin\aleph(t_k)}e_i^T(t_k^-)Pe_i(t_k^-)\\&\le \mathfrak{b}'_{1k}V(t_k^-)+\mathfrak{b}_{2k}[V(t_k^-)]_{\tau_1}+\mathfrak{b}_{3k}[V(t_k^-)]_{\tau_2}\\&\quad+\mathfrak{b}_{4k}[V(t_k^-)]_{-\infty}+\mathfrak{b}_{5k}[V(t_k^-)]_{\tau_3}\\&\quad+\frac{N-l_k}{N}[V(t_k^-)]\\&= \mathfrak{b}_{1k}V(t_k^-)+\mathfrak{b}_{2k}[V(t_k^-)]_{\tau_1}+\mathfrak{b}_{3k}[V(t_k^-)]_{\tau_2}\\&\quad+\mathfrak{b}_{4k}[V(t_k^-)]_{-\infty}+\mathfrak{b}_{5k}[V(t_k^-)]_{\tau_3} \qquad (37)\end{aligned}$$

where

$$\begin{aligned}\mathfrak{b}'_{1k} &= \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\bigg((1+\varepsilon_1)(1+\gamma_k)^2\frac{l_k}{N}\\&\qquad+(1+\varepsilon_1^{-1})(1+\xi_1)\Xi\bigg)\\\mathfrak{b}_{1k} &= 1-\frac{l_k}{N}+\mathfrak{b}'_1\\\mathfrak{b}_{2k} &= \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(1+\varepsilon_1^{-1})(1+\xi_1)\Xi_1\\\mathfrak{b}_{3k} &= \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(1+\varepsilon_1^{-1})(1+\xi_1)\Xi_2\end{aligned}$$

$$\mathfrak{b}_{4k} = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(1+\varepsilon_1^{-1})(1+\xi_1)\Xi_3$$
$$\mathfrak{b}_{5k} = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(1+\varepsilon_1^{-1})(1+\xi_1^{-1})\gamma_k^4\varsigma^2\frac{l_k}{N}$$

Then

$$\begin{aligned}&\mathbb{E}\{V(t_k^+)\}\\ &\le \mathfrak{b}_{1k}\mathbb{E}V(t_k^-) + \mathfrak{b}_{2k}\mathbb{E}[V(t_k^-)]_{\tau_1} + \mathfrak{b}_{3k}\mathbb{E}[V(t_k^-)]_{\tau_2}\\ &\quad + \mathfrak{b}_{4k}\mathbb{E}[V(t_k^-)]_{-\infty} + \mathfrak{b}_{5k}\mathbb{E}[V(t_k^-)]_{\tau_3}\end{aligned} \tag{38}$$

By Lemma 3 , it follows from (29) to (38) hold, we have

$$\mathbb{E}\{V(t)\} \le \chi e^{-\eta t}, \quad t \ge 0. \tag{39}$$

where $\chi = \lambda_{\max}(P)M\mathbb{E}\sum_{i=1}^{N}\sup_{-\tau\le s\le 0}\{\|\overline{\phi}_i(s)\|^2\}, \eta > 0$, $M > 1$. Thus this completes the proof. □

Remark 7 If the delay of impulsive controller $\tau_3(t) = 0$, the controller will reduce to the following impulsive pinning controller.

$$u_i(t) = \begin{cases}\sum_{k=1}^{+\infty}\gamma_k e_i(t)\delta(t-t_k^-), & i \in \aleph_k\\ 0, & i \notin \aleph_k\end{cases} \tag{40}$$

If $l_k = N$, all the agents will be controlled at each impulsive instant, then the delayed impulsive pinning controller reduces to general delayed impulsive controller

$$u_i(t) = \sum_{k=1}^{+\infty}\gamma_k e_i(t-\tau_3(t))\delta(t-t_k^-), \tag{41}$$

Furthermore, for both $l_k = N$ and $\tau_3(t) = 0$, we can get the standard linear impulsive controller

$$u_i(t) = \sum_{k=1}^{+\infty}\gamma_k e_i(t)\delta(t-t_k^-), \tag{42}$$

Thus it is noted that our proposed delayed impulsive pinning controller is more general.

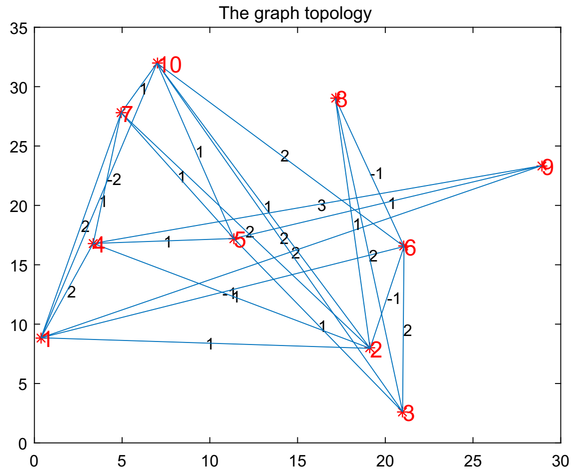

Fig. 1 The topology of corresponding coupling matrix L

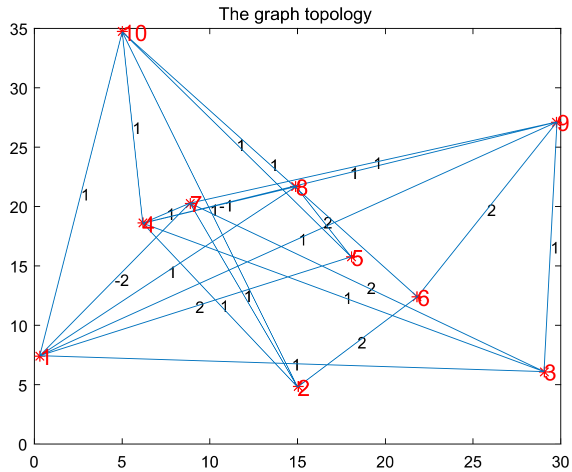

Fig. 2 The topology of corresponding coupling matrix H

4 Simulation results

Consider the fixed undirected interaction topologies shown in Figs. 1 and 2, which are consisting of 10 agents.

The parameters of multi-agent systems (1) and (2) are

$$\mathcal{D} = \begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}, \mathcal{A} = \begin{bmatrix}0.3 & -0.3\\ 0.4 & 0.5\end{bmatrix},$$
$$\mathcal{B} = \begin{bmatrix}-1.4 & 0.1\\ 0.3 & -0.9\end{bmatrix},$$

Table 1 The relationship of the number of pinned agents and the upper bound of impulsive interval

l_k	1	2	3	4	5	6	7	8	9	10
$\Delta(t_k)$	0.0058	0.0145	0.0244	0.0355	0.0482	0.0632	0.0813	0.1042	0.1347	0.1765

The coupling matrices L and H are

$$L=\begin{bmatrix} -7 & 1 & 0 & 2 & 0 & -1 & 2 & 0 & 2 & 1\\ 0 & -5 & 0 & 1 & 0 & -1 & 2 & 1 & 0 & 1\\ -2 & 1 & -6 & 0 & 1 & 2 & 0 & 2 & 0 & 2\\ 1 & 1 & 0 & -4 & 1 & 0 & -2 & 0 & 3 & 0\\ 0 & 1 & -1 & 0 & -3 & 0 & 1 & 0 & 1 & 1\\ 0 & 1 & 2 & 1 & 0 & -5 & 0 & -1 & 0 & 2\\ 2 & 0 & 1 & 2 & 0 & 2 & -8 & 0 & 0 & 1\\ 0 & -2 & 3 & 0 & 2 & 0 & 1 & -4 & 0 & 0\\ 1 & 0 & 3 & 0 & -1 & 1 & 0 & 0 & -4 & 0\\ 0 & 1 & 0 & 2 & -1 & 0 & 1 & 3 & 1 & -7 \end{bmatrix},$$

$$H=\begin{bmatrix} -4 & 0 & 1 & 0 & 2 & 0 & -2 & 1 & 1 & 1\\ 0 & -5 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1\\ -1 & 0 & -3 & 1 & 0 & 0 & 2 & 0 & 1 & 0\\ 0 & 0 & 1 & -3 & 0 & 0 & 1 & -1 & 1 & 1\\ 2 & 0 & 2 & 0 & -7 & 0 & 0 & 2 & 0 & 1\\ 1 & 0 & 0 & 1 & 1 & -6 & 0 & 0 & 2 & 1\\ 0 & 1 & 0 & 1 & 0 & 1 & -4 & 0 & 1 & 0\\ 1 & 0 & 0 & 3 & 1 & 0 & -1 & -4 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & -5 & 0\\ 0 & 1 & 0 & 1 & -2 & 0 & 0 & 2 & 1 & -3 \end{bmatrix}.$$

$c_1 = 0.2, c_2 = 0.5, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0.1$, $\gamma_{1k} = -0.8, \gamma_{2k} = 0.05, \theta = 0.6, K(t) = e^{-0.5t}$. The time delays are chosen as

$$\tau_1(t) = \frac{e^t}{1+e^t}, \quad \tau_2(t) = \frac{(1+\cos(t))e^t}{1+e^t},$$
$$\tau_3(t) = \frac{1+0.2\sin^2(t)}{0.5+e^t},$$

and

$$f(x_i(t)) = 0.5(\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^T.$$
$$g(x_i(t), x_i(t-\tau_2(t)), t) = 0.3|x_i(t)|I_2.$$

$\rho_1 = \rho_2 = 0.09, \mu = 0.25.\ P = 0.5I_2.$

Set the initial value as

$$x_i(0) = [3+0.5i, 2-1i]^T,$$
$$s(0) = [3.5, 0.5]^T, i = 1, 2, \ldots, 10.$$

By a simple computation, the number of pinned agents l_k and an upper bound for the impulsive interval $\Delta(t_k) = \max\{t_{k+1} - t_k\}$ can be obtained, which are presented in Table 1.

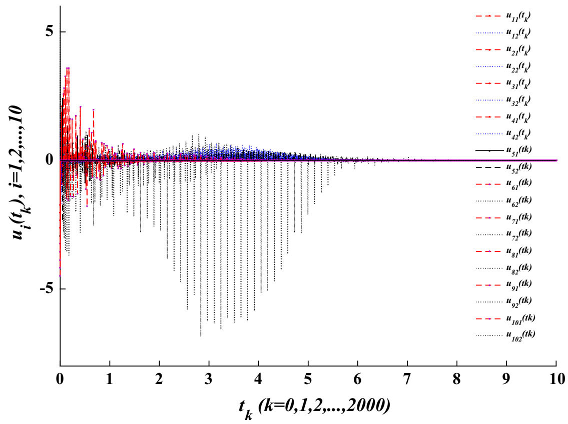

Fig. 3 The trajectories of impulsive pinning controllers

It can be seen that as the number of pinned agents increases, the length of the impulsive interval increases as well, which is in consistent with our common sense. Thus, if the pinning ratio is set to $\frac{l_k}{N} = 0.5$, the impulsive interval should satisfy that $t_{k+1} - t_k < 0.0482$. Setting the impulsive interval to $t_{k+1} - t_k = 0.04$, then we have $\lambda_{\max}(L) = -0.0587, \mu = 0.25, \mathfrak{b}'_{1k} = 0.0440, \mathfrak{b}_{1k} = 0.5220, \mathfrak{b}_{2k} = 0.0275$, $\Pi = \begin{bmatrix} 5.3423 & -1.3500 \\ -1.3500 & 3.3323 \end{bmatrix}$, $\lambda_{\max}(\Pi) = 6.0203, \bar{k} = 1, p = 12.0405, q_1 = 0.0500, q_2 = 0.1080, q_3 = 0.0500$. Then the conditions of Theorem 1 are satisfied

$$\mathfrak{b}_{1k} + \mathfrak{b}_{2k} = 0.5495 < 1, \tag{43}$$

$$p + \frac{\sum_{i=1}^{3} q_i}{\mathfrak{b}_{1k} + \mathfrak{b}_{2k}} + \frac{ln(\mathfrak{b}_{1k} + \mathfrak{b}_{2k})}{t_{k+1} - t_k} = -2.5496 < 0 \tag{44}$$

To choose the agents which agents should be pinned, we can use this effective way. At each impulsive time instant t_k, rearrange the error states of agents such that $\|e_{i1}(t_k)\| \geq \|e_{i2}(t_k)\| \geq \cdots \geq \|e_{il}(t_k)\| \geq \|e_{i,l+1}(t_k)\| \geq \cdots \geq \|e_{iN}(t_k)\|$. Choose the first l_k nodes as controlled agents. Figure 3 describes the trajectories of impulsive pinning controllers.

Figures 4 and 5 show the trajectories of systems without and with controllers, respectively. It can be

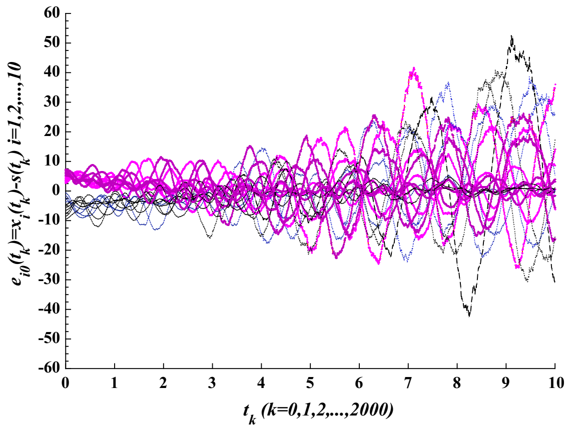

Fig. 4 The error states trajectories of the multi-agent systems without controller

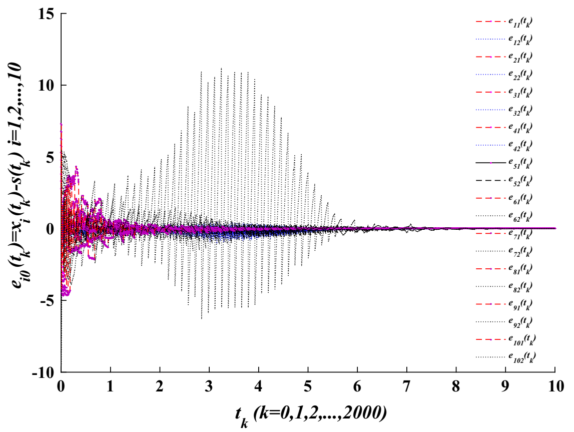

Fig. 5 The error states trajectories of the multi-agent systems with impulsive pinning controller

seen that the proposed strategy is effective for stochastic multi-agent systems.

5 Conclusions

In this technical note, the mean square exponential consensus problem was analyzed. Two novel delayed impulsive pinning control strategies were proposed that are different from general delay-free impulsive pinning control algorithms. In terms of Halanay inequality technique, two criteria were established. In addition, the trade-off between the number of pinned agents and the upper bound of impulsive interval was discussed. In future work, impulsive control should be further investigated in other circumstances. There are several possible extensions to be worthy of further exploring. For examples, the H_∞ filtering problem for semi-Markov jump systems [31] and finite-time synchronization [32] have attracted a great deal of attention. The extension of the proposed results to these problems appears to be a challenging problem.

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